Math 424 – Prof. Richard B. Goldstein – Chapter 7 HW – 7th Edition

#2 7.2 From $y = x^2$, $x = 0, 1, 2, 3$, we obtain $x = \sqrt{y}$,

$$g(y) = f(\sqrt{y}) = \binom{3}{\sqrt{y}} \left(\frac{2}{5}\right)^{\sqrt{y}} \left(\frac{3}{5}\right)^{3-\sqrt{y}}, \qquad \text{for } y = 0, 1, 4, 9.$$

#4 7.4

$$P(Y = 1) = f(1, 1) = 1/18,$$
$$P(Y = 2) = f(1, 2) + f(2, 1) = 2/18 + 2/18 = 2/9,$$
$$P(Y = 3) = f(1, 3) = 3/18 = 1/6,$$
$$P(Y = 4) = f(2, 2) = 4/18 = 2/9,$$
$$P(Y = 6) = f(2, 3) = 6/18 = 1/3.$$

#6 7.6 The inverse function of $y = 8x^3$ is $x = y^{1/3}/2$, for $0 < y < 8$ from which we obtain $|J| = y^{-2/3}/6$. Therefore,

$$g(y) = f(y^{1/3}/2)|J| = 2(y^{1/3}/2)(y^{-2/3}/6) = \frac{1}{6}y^{-1/3}, \quad 0 < y < 8.$$

#9 7.9 (a) The inverse of $y = x + 4$ is $x = y - 4$, for $y > 4$, from which we obtain $|J| = 1$. Therefore,

$$g(y) = f(y - 4)|J| = 32/y^3, \quad y > 4.$$

(b) $P(Y > 8) = 32 \int_8^\infty y^{-3}\, dy = -\ 16y^{-2}\big|_8^\infty = \frac{1}{4}$.

#12 7.12 Since X_1 and X_2 are independent, the joint probability distribution is

$$f(x_1, x_2) = f(x_1)f(x_2) = e^{-(x_1+x_2)}, \quad x_1 > 0,\ x_2 > 0.$$

The inverse functions of $y_1 = x_1 + x_2$ and $y_2 = x_1/(x_1 + x_2)$ are $x_1 = y_1 y_2$ and $x_2 = y_1(1 - y_2)$, for $y_1 > 0$ and $0 < y_2 < 1$, so that

$$J = \begin{vmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1.$$

Then, $g(y_1, y_2) = f(y_1y_2, y_1(1-y_2))|J| = y_1e^{-y_1}$, for $y_1 > 0$ and $0 < y_2 < 1$. Therefore,

$$g(y_1) = \int_0^1 y_1 e^{-y_1}\, dy_2 = y_1 e^{-y_1}, \quad y_1 > 0,$$

and

$$g(y_2) = \int_0^\infty y_1 e^{-y_1}\, dy_1 = \Gamma(2) = 1, \quad 0 < y_2 < 1.$$

Since $g(y_1, y_2) = g(y_1)g(y_2)$, the random variables Y_1 and Y_2 are independent.

#14 7.14 The inverse functions of $y = x^2$ are given by $x_1 = \sqrt{y}$ and $x_2 = -\sqrt{y}$ from which we obtain $J_1 = 1/2\sqrt{y}$ and $J_2 = 1/2\sqrt{y}$. Therefore,

$$g(y) = f(\sqrt{y})|J_1| + f(-\sqrt{y})|J_2| = \frac{1+\sqrt{y}}{2} \cdot \frac{1}{2\sqrt{y}} + \frac{1-\sqrt{y}}{2} \cdot \frac{1}{2\sqrt{y}} = 1/2\sqrt{y},$$

for $0 < y < 1$.

#17 7.17 The moment-generating function of X is

$$M_X(t) = E(e^{tX}) = \frac{1}{k}\sum_{x=1}^{k} e^{tx} = \frac{e^t(1-e^{kt})}{k(1-e^t)},$$

by summing the geometric series of k terms.

#18 7.18 The moment-generating function of X is

$$M_X(t) = E(e^{tX}) = p\sum_{x=1}^{\infty} e^{tx}q^{x-1} = \frac{p}{q}\sum_{x=1}^{\infty}(e^t q)^x = \frac{pe^t}{1-qe^t},$$

by summing an infinite geometric series. To find out the moments, we use

$$\mu = M_X'(0) = \left.\frac{(1-qe^t)pe^t + pqe^{2t}}{(1-qe^t)^2}\right|_{t=0} = \frac{(1-q)p + pq}{(1-q)^2} = \frac{1}{p},$$

and

$$\mu_2' = M_X''(0) = \left.\frac{(1-qe^t)^2pe^t + 2pqe^{2t}(1-qe^t)}{(1-qe^t)^4}\right|_{t=0} = \frac{2-p}{p^2}.$$

So, $\sigma^2 = \mu_2' - \mu^2 = \frac{q}{p^2}$.