Bivariate Distribution Examples – Prof. Richard B. Goldstein

Discrete

height\weight	125 to 150	150 to 175	175 to 200	200 to 225	225 to 250	
4'6" to 5'0"	0.02	0.04	0.02	0.01		0.09
5'0" to 5'6"	0.02	0.04	0.10	0.03	0.01	0.20
5'6" to 6'0"	0.03	0.18	0.30	0.10	0.01	0.62
6'0" to 6'6"			0.03	0.04	0.02	0.09
	0.07	0.26	0.45	0.18	0.04	1.00

Let X = height and Y = weight

The marginal distributions are shown as row sums and columns sums. The joint distribution is not independent – for example $P(5'0'' < X < 5'6'')P(175 < Y < 200) \neq P(5'0'' < X < 5'6'' \text{ and } 175 < Y < 200)$. That is $0.20 \times 0.45 = 0.09 \neq 0.10$.

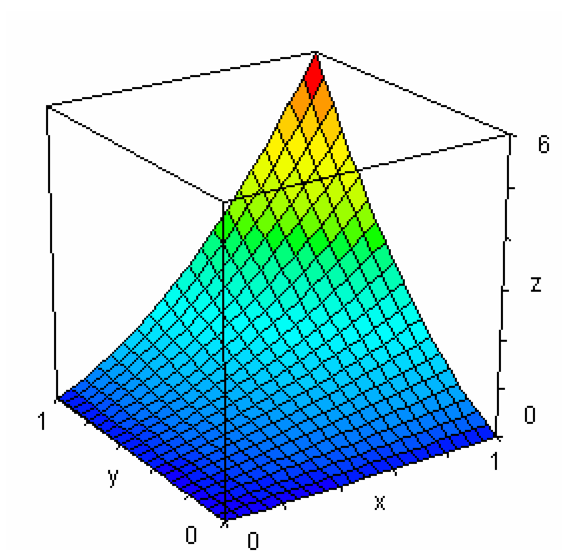
Conditional Probability for Y | X is between 5’0” and 5’6”

125 to 150	150 to 175	175 to 200	200 to 225	225 to 250
0.02/0.20 = 0.10	0.04/0.20 = 0.20	0.10/0.20 = 0.50	0.03/0.20 = 0.15	0.01/0.20 = 0.05

$$E(Y|5'0'' < X < 5'6'') = 137.5(0.10) + 162.5(0.20) + 187.5(0.50) + 212.5(0.15) + 237.5(0.05)$$
$$= 13.75 + 32.50 + 93.75 + 31.875 + 11.875 = 183.75$$

Continuous

$f(x, y) = 2.4xy + 3.6x^2y^2$ for $0 \le x \le 1,\ 0 \le y \le 1$



$$g(x) = \int_0^1 2.4xy + 3.6x^2y^2\,dy = 1.2x + 1.2x^2$$

$$h(y) = \int_0^1 2.4xy + 3.6x^2y^2\,dx = 1.2y + 1.2y^2$$

$$f(x\,|\,y) = \frac{f(x,y)}{h(y)} = \frac{2.4xy + 3.6x^2y^2}{1.2y + 1.2y^2} = \frac{2x + 3x^2y}{1+y}$$

For example, if $y = 0.6$ then $f(x \mid y) = 1.25x + 1.125x^2$

Likewise, $$f(y\,|\,x) = \frac{f(x,y)}{g(x)} = \frac{2.4xy + 3.6x^2y^2}{1.2x + 1.2x^2} = \frac{2x + 3x^2y}{1+x}$$

Note that $f(x, y) \neq g(x)h(y)$ and therefore X and Y are dependent. If we used $f(x, y) = 4xy$ then we would have X and Y independent.

Note that $\int_0^1 f(x\mid y)\,dx = \int_0^1 \frac{2x+3x^2y}{1+y}\,dx = \left.\frac{x^2+x^3y}{1+y}\right|_0^1 = \frac{1+y}{1+y} - \frac{0}{1+y} = 1$

To find the expected value for the conditional case,

$$E(X\mid Y) = \int_0^1 x\frac{2x+3x^2y}{1+y}\,dx = \int_0^1 \frac{2x^2+3x^3y}{1+y}\,dx = \left.\frac{\frac{2x^3}{3}+\frac{3x^4y}{4}}{1+y}\right|_0^1 = \frac{\frac{2}{3}+\frac{3y}{4}}{1+y} - 0 = \frac{8+9y}{12(1+y)}$$

If Y = 0.6, E(X | Y = 0.6) = 13.4/19.2 = 0.69791666… which is the same result as

$$\int_0^1 x\left(1.25x+1.125x^2\right)dx = \int_0^1 1.25x^2+1.125x^3\,dx = \left.\frac{1.25x^3}{3}+\frac{1.125x^4}{4}\right|_0^1 = 0.69791666... = \frac{67}{96}$$