Padé - Rational Polynomial Approximations – Prof. Richard B. Goldstein

Approximate: $f(x) = a_0 + a_1 x + \cdots + a_N x^N + \cdots$ by $R_{m,n}(x) = \dfrac{p_0 + p_1 x + \cdots + p_m x^m}{q_0 + q_1 x + \cdots + q_n x^n}$ where N = m + n

To find the p's and q's (assume $q_0 = 1$) and rearrange: $f(x) \approx \dfrac{P_m(x)}{Q_n(x)} \Rightarrow \dfrac{f(x)Q_n(x) - P_m(x)}{Q_n(x)} \approx 0$

in which the numerator's coefficients of 1, x, …, x^{m+n} are set to 0. This will result in a system of m linear equations in m unknowns involving the q's alone and simple equations for the p's.

Example #1 $f(x) = \tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

is accurate to within ±0.0133 on [-π/4, π/4]. Using **m = 3** and **n = 2**

$$\frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3}{1 + q_1 x + q_2 x^2} \approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

$$p_0 + p_1 x + p_2 x^2 + p_3 x^3 \approx \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5\right)\left(1 + q_1 x + q_2 x^2\right)$$

<u>Equate</u> the coefficients on both sides:

1:	p_0	=	0
x:	p_1	=	1
x^2:	p_2	=	q_1
x^3:	p_3	=	$\frac{1}{3} + q_2$
x^4:	0	=	q_1
x^5:	0	=	$\frac{2}{15} + \frac{q_2}{3}$

<u>Solve</u> the last two equations first and substitute the q_1 and q_2 to find the p's

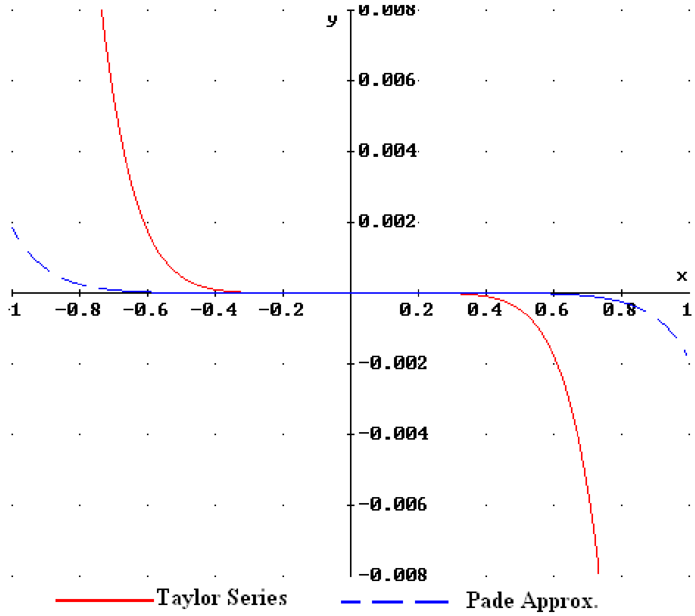
$$q_1 = 0,\ q_2 = -\frac{2}{5},\ p_0 = 0,\ p_1 = 1,\ p_2 = 0,\ p_3 = -\frac{1}{15}$$

$f(x) \approx \dfrac{x - \frac{x^3}{15}}{1 - \frac{2x^2}{5}}$ is accurate to within ±0.000212 on [-π/4, π/4]

<u>Note</u>: Using the Taylor expansion of sin(x) and cos(x) one would expect

$f(x) \approx \dfrac{x - \frac{x^3}{6}}{1 - \frac{x^2}{2}}$ but that is accurate to within ±0.0189 on [-π/4, π/4]

Error Curves



If the approximation is repeated with **m = 5** and **n = 4,** noting the various coefficients that are zero:

$$\frac{p_1 x + p_3 x^3 + p_5 x^5}{1 + q_2 x^2 + q_4 x^4} \approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots$$

$$p_1 x + p_3 x^3 + p_5 x^5 + p_3 x^3 \approx \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 \right)\left(1 + q_2 x^2 + q_4 x^4\right)$$

<u>Equate</u> the coefficients on both sides:

x:	p_1	=	1
x^3:	p_3	=	$q_2 + \frac{1}{3}$
x^5:	p_2	=	$q_4 + \frac{q_2}{3} + \frac{2}{15}$
x^7:	0	=	$\frac{q_4}{3} + \frac{2q_2}{15} + \frac{17}{315}$
x^9:	0	=	$\frac{2q_4}{15} + \frac{17q_2}{315} + \frac{62}{2835}$

<u>Solve</u> the last two equations first and substitute the q_2 and q_4 to find the p's

$$q_2 = -\frac{4}{9},\ q_4 = \frac{1}{63},\ p_1 = 1,\ \ p_3 = -\frac{1}{9},\ \ p_5 = \frac{1}{945}$$

$$f(x) \approx \frac{x - \frac{x^3}{9} + \frac{x^5}{945}}{1 - \frac{4x^2}{9} + \frac{x^4}{63}}$$ is accurate to within $\pm 1.347 \times 10^{-8}$ on $[-\pi/4, \pi/4]$

<u>Note</u>: The terms in the numerator and denominator are getting closer to the Taylor Series terms for sin(x) and cos(x)

Example #2

$$f(x) = \ln\left(\frac{1+0.8x}{1-0.2x}\right) = x - \frac{3}{10}x^2 + \frac{13}{75}x^3 - \frac{51}{500}x^4 + \frac{41}{625}x^5 + \cdots$$

is accurate within ±0.02600 on [0, 1]. Using **m = 3** and **n = 2**

$$\frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3}{1 + q_1 x + q_2 x^2} \approx x - \frac{3}{10}x^2 + \frac{13}{75}x^3 - \frac{51}{500}x^4 + \frac{41}{625}x^5 + \cdots$$

$$p_0 + p_1 x + p_2 x^2 + p_3 x^3 \approx \left(x - \frac{3}{10}x^2 + \frac{13}{75}x^3 - \frac{51}{500}x^4 + \frac{41}{625}x^5\right)\left(1 + q_1 x + q_2 x^2\right)$$

Equate the coefficients on both sides:

1:	p_0	=	0
x:	p_1	=	1
x^2:	p_2	=	$-\frac{3}{10} + q_1$
x^3:	p_3	=	$\frac{13}{75} - \frac{3q_1}{10} + q_2$
x^4:	0	=	$-\frac{51}{500} + \frac{13q_1}{75} - \frac{3q_2}{10}$
x^5:	0	=	$\frac{41}{625} - \frac{51q_1}{500} + \frac{13q_2}{75}$

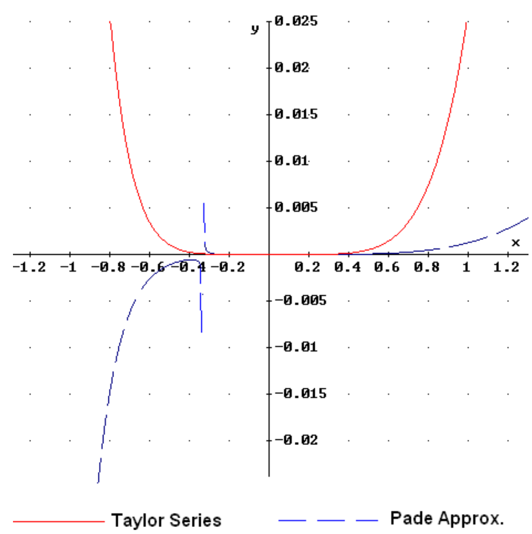
Solve the last two equations first and substitute the q_1 and q_2 to find the p's

$$q_1 = \frac{18}{5},\ q_2 = \frac{87}{50},\ p_0 = 0,\ p_1 = 1,\ p_2 = \frac{33}{10},\ p_3 = \frac{5}{6}$$

$$f(x) \approx \frac{x + 3.3x^2 + 0.8333333...x^3}{1 + 3.6x + 1.74x^2}$$ is accurate within ±0.00126 on [0, 1]

Note: Since the denominator in the Padé approximation has a root at -0.33, the approximation is only good for x > 0.

Error Curves



Continued Fraction Expansions

$$\frac{2x^2+22x+58}{x^3+14x^2+60x+73}=\frac{(2x+22)x+58}{((x+14)x+60)x+73}=\cfrac{2}{x+3-\cfrac{2}{x+4+\cfrac{1}{x+7}}}$$

The first expression 8 $\times$, 5 $\pm$, and 1 $\div$ or 14 ops; the second requires 4 $\times$, 5 $\pm$, and 1 $\div$ or 10 ops; and the third requires 5 $\pm$ and 3 $\div$ or only 8 ops.

$$\frac{2x^2+22x+58}{x^3+14x^2+60x+73}=\frac{2(x^2+11x+29)}{x^3+14x^2+60x+73}=\cfrac{2}{\cfrac{x^3+14x^2+60x+73}{x^2+11x+29}}$$

$$=\cfrac{2}{x+3-\cfrac{2x+14}{x^2+11x+29}}=\cfrac{2}{x+3-\cfrac{2(x+7)}{x^2+11x+29}}=\cfrac{2}{x+3-\cfrac{2}{\cfrac{x^2+11x+29}{x+7}}}$$

$$=\cfrac{2}{x+3-\cfrac{2}{x+4+\cfrac{1}{x+7}}}$$

Algebraic long-division steps:

$$\begin{array}{r|l} & x+3 \\ \hline x^2+11x+29 & x^3+14x^2+60x+73 \\ & \underline{x^3+11x^2+29x} \\ & \quad 3x^2+31x+73 \\ & \quad \underline{3x^2+33x+87} \\ & \qquad -2x-14 \end{array}$$

and

$$\begin{array}{r|l} & x+4 \\ \hline x+7 & x^2+11x+29 \\ & \underline{x^2+7x} \\ & \quad 4x+29 \\ & \quad \underline{4x+28} \\ & \qquad 1 \end{array}$$