MTH/CSC 440 – Prof. Richard B. Goldstein – Chap 8 HW – B+F 8th Ed

Chap 8 – Approximation Theory

8.1 #3

x_i	y_i
1	1.84
1.1	1.96
1.3	2.21
1.5	2.45
1.9	2.94
2.1	3.18

$n = 6$

$\sum x = 8.9$ $\qquad \sum y = 14.58$

$\sum x^2 = 14.17$ $\qquad \sum xy = 22.808$

$\sum x^3 = 24.023$ $\qquad \sum x^2 y = 38.0962$

$\sum x^4 = 42.8629$

LINEAR

Solve $\left.\begin{array}{l} 6a_0 + 8.9a_1 = 14.58 \\ 8.9a_0 + 14.17a_1 = 22.808 \end{array}\right\} \Rightarrow \hat{y} = 0.620895 + 1.219621x$

$\hat{y}_i$: 1.840516, 1.962478, 2.206402, 2.450327, 2.938175, 3.182099

$\sum(\hat{y}_i - y_i)^2 \cong 2.719 \times 10^{-5}$

QUADRATIC

Solve $\left.\begin{array}{l} 6a_0 + 8.9a_1 + 14.17a_2 = 14.58 \\ 8.9a_0 + 14.17a_1 + 24.023a_2 = 22.808 \\ 14.17a_0 + 24.023a_1 + 42.8629a_2 = 38.0962 \end{array}\right\} \Rightarrow \begin{array}{l} \hat{y} = 0.596581 + 1.253293x \\ \quad -0.0108534x^2 \end{array}$

$\hat{y}_i$: 1.839021, 1.962071, 2.207520, 2.452100, 2.938657, 3.180633

$\sum(\hat{y}_i - y_i)^2 \cong 1.801 \times 10^{-5}$

EXTRA

8.1 #5d $\quad y = be^{ax} \Rightarrow \ln y = \ln b + ax = B + ax$ where $b = e^B$

now we need to solve : $\underset{n}{10}B + \underset{\sum x}{54.1}\,a = \underset{\sum \ln y}{52.033632}$

$54.1B + \underset{\sum x^2}{303.39}\,a = \underset{\sum x \ln y}{285.48979}$

gives $a = 0.3723817$, $b = e^B = e^{3.188778} = 24.258761$

$\hat{y} = 24.258761\,e^{0.3723817x}$ but sum sq = 41.769 (not a good fit)

EXTRA

* 8.1 # 5a $\quad y = bx^a \Rightarrow \ln y = \underbrace{\ln b}_{=B} + a \ln x$

now solve: $\quad n,\ \Sigma \ln x,\ \Sigma \ln y$

$$10\,B + 16.69952\,a = 52.033632$$

$$16.69952\,B + 28.253712\,a = 87.633445$$

$\Sigma(\ln x)^2 \qquad \Sigma(\ln x)(\ln y)$

$B = 1.8308245 \qquad b = e^B = 6.2390287$

$a = 2.0195414$

or $\hat{y} = 6.2390287\, x^{2.0195414}$ $\qquad$ sum sqs $= 7.02 \times 10^{-3}$

Sect 8.2

#1c $\quad f(x) = \frac{1}{x}$ on $[1,3]$

NORMAL EQS

$$\left.\begin{aligned} a_0 \int_1^3 1\,dx + a_1 \int_1^3 x\,dx &= \int_1^3 (1)\tfrac{1}{x}\,dx \\ a_0 \int_1^3 x\,dx + a_1 \int_1^3 x^2\,dx &= \int_1^3 x\left(\tfrac{1}{x}\right)dx \end{aligned}\right\} \Rightarrow \begin{aligned} 2a_0 + 4a_1 &= \underbrace{1.098612}_{\ln 3} \\ 4a_0 + \tfrac{26}{3}a_1 &= 2 \end{aligned}$$

solution: $\quad a_0 + a_1 x = \underline{1.1409799 - 0.2958369\,x}$

#3c NORMAL EQS

$$\left.\begin{aligned} 2a_0 + 4a_1 + \tfrac{26}{3}a_2 &= 1.098612 \\ 4a_0 + \tfrac{26}{3}a_1 + 20a_2 &= 2 \\ \tfrac{26}{3}a_0 + 20a_1 + \tfrac{242}{5}a_2 &= 4 \end{aligned}\right\} \Rightarrow \quad a_0 + a_1 x + a_2 x^2$$

$$= 1.7235344 - 0.9313509\,x + 0.1588785\,x^2$$

Sect 8.2

#2c $f(x) = \frac{1}{x+2}$ on $[-1,1]$

$P_0(x) = 1$

$P_1(x) = x$

$$a_0 = \frac{1}{(2/1)} \int_{-1}^{1} (1) \frac{1}{x+2}\,dx$$

$$= \frac{1}{2} \ln(x+2)\Big|_{-1}^{+1} = \frac{1}{2}\ln 3 = 0.5493061$$

(note: $\int_{-1}^{1} P_k^2(x)\,dx = \frac{2}{2k+1}$)

$$a_1 = \frac{1}{(2/3)} \int_{-1}^{1} x \left(\frac{1}{x+2}\right) dx = \frac{3}{2} \int_{-1}^{1} \left(1 - \frac{2}{x+2}\right) dx$$

$$= \frac{3}{2}\left(x - 2\ln(x+2)\right)\Big|_{-1}^{1} = \frac{3}{2}(1 - 2\ln 3) - \frac{3}{2}(-1)$$

$$= -0.29583669$$

or $P_1(x) = 0.5493061 - 0.2958369\,x$ (Same as 1(c) if $x \to x-2$)

#4c – continue from part #4c

$P_2(x)$ standard form $P_k(1) = 1$

$$a_2 = \frac{1}{2/5} \int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) \frac{1}{x+2}\,dx = \frac{5}{2} \int_{-1}^{1} \frac{3x^2 - 1}{2x+4}\,dx$$

$$= \frac{5}{2} \int_{-1}^{1} \left(\frac{3}{2}x - 3 + \frac{11}{2x+4}\right) dx = \frac{5}{2}\left(\frac{3x^2}{4} - 3x + \frac{11}{2}\ln(x+2)\right)\Big|_{-1}^{1}$$

$$= \frac{5}{2}\left(\frac{3}{4} - 3 + \frac{11}{2}\ln 3\right) - \frac{5}{2}\left(\frac{3}{4} + 3 + 0\right) = 0.105919$$

$$P_2(x) = 0.5493061 - 0.2958369\,x + 0.105919\left(\frac{3x^2}{2} - \frac{1}{2}\right)$$

$$= 0.4963466 - 0.2958369\,x + 0.1588785\,x^2$$

(also same as #2c if $x \to x-2$)

Sect 8.3

#1c $f(x) = \ln(x+2)$ on $[-1,1]$

$\tilde{T}_3(x)$ has roots $\cos\frac{\pi}{6}, \cos\frac{3\pi}{6}, \cos\frac{5\pi}{6}$ or $\pm\frac{\sqrt{3}}{2}, 0$

x	$f(x)$	1st DD	2nd DD
$-\sqrt{3}/2$	$\ln(1.1339746) = 0.1257288$		
		> 0.6551483	
0	$\ln(2) = 0.6931472$		> -0.1384263
		> 0.4154370	
$\sqrt{3}/2$	$\ln(2.8660254) = 1.0529262$		

$P_2(x) = 0.1257288 + 0.6551483(x+0.8660254) - 0.1384263(x+0.8660254)(x-0)$

$= \boxed{-0.1384263x^2 + 0.5353176x + 0.6931472}$

#5a $f(x) = \frac{1}{x}$ on $[1,3]$ $\Rightarrow f^*(x) = \frac{1}{x+2}$ on $[-1,1]$

x	$f^*(x)$	1st DD	
$-\sqrt{3}/2$	0.8818540		
		> -0.4409270	
0	0.5000000		> 0.1538462
		> -0.1794576	
$\sqrt{3}/2$	0.3489153		

$P_2(x) = 0.8818540 - 0.4409270(x+0.8660254) + 0.1538462(x+0.8660254)(x-0)$

$= \boxed{0.5 - 0.3076923x + 0.1538462x^2}$ for $\frac{1}{x+2}$

replace $x \to x-2$

#6 $f(x) = xe^x$ keep error within 0.01 on $[-1,1]$

$f'(x) = (x+1)e^x$ $f(0)=0$ $f'(0)=1$ $f''(0)=2$ ... $f^{(n)}(0)=n$

$f''(x) = (x+2)e^x$

Maclaurin Series :

$\vdots$

$f^{(n)}(x) = (x+n)e^x$

$P_6(x) = 0 + 1x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 + \frac{5}{5!}x^5 + \frac{6}{6!}x^6$

$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}$

$f^{(7)}(x) = (x+7)e^x$. on $[-1,1]$ has a max at $c=1$ $f^{(7)}(1) = 8e$

$\therefore$ Max Trunc. error $= \frac{8e}{7!}(1)^7 = 0.0043147$

#6 CONTINUED

$$P_6 = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}$$

$$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{1}{24}\left(\frac{1}{16}\right)(10T_1 + 5T_3 + T_5) + \frac{1}{120}\left(\frac{1}{32}\right)(10T_0 + 15T_2 + 6T_4 + T_6)$$

$$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{1}{384}\left(10x + 5(4x^3 - 3x)\right) + \frac{1}{3840}\left(10 + 15(2x^2-1) + 6(8x^4 - 8x^2 + 1)\right) + \underbrace{\frac{T_5}{384} + \frac{T_6}{3840}}$$

by dropping the last two terms error increases to:

$$\text{ERROR} \le 0.0043147 + \left|\frac{1}{384}\right| + \left|\frac{1}{3840}\right| = \underline{\underline{0.007179}}$$

now

$$P_4(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{5}{384}x + \frac{20}{384}x^3 + \frac{1}{3840} - \frac{18}{3840}x^2 + \frac{48}{3840}x^4$$

$$= \boxed{\left(\frac{1}{3840}\right) + \left(\frac{379}{384}\right)x + \left(\frac{637}{640}\right)x^2 + \left(\frac{53}{96}\right)x^3 + \left(\frac{43}{240}\right)x^4}$$

$$= \boxed{0.00026042 + 0.9869792x + 0.9953125x^2 + 0.5508333x^3 + 0.1791667x^4 \quad \text{with error} \le 0.007179}$$

approximates xe^x on $[-1,1]$

Sect 8.4 #6

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$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots$$

$n=3 \quad m=3$

$$\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)\left(1 + q_1 x + q_2 x^2 + q_3 x^3\right) - \left(p_0 + p_1 x + p_2 x^2 + p_3 x^3\right)$$

q_0

coeffs

$$\left.\begin{aligned} x^6&: \tfrac{1}{120} q_1 - \tfrac{1}{6} q_3 = 0 \\ x^5&: \tfrac{1}{120} - \tfrac{q_2}{6} = 0 \\ x^4&: -\tfrac{q_1}{6} + q_3 = 0 \end{aligned}\right\} \Rightarrow q_1 = 0 \quad q_2 = \frac{1}{20} \quad q_3 = 0$$

$$\left.\begin{aligned} x^3&: -\tfrac{1}{6} + q_2 - p_3 = 0 \\ x^2&: q_1 - p_2 = 0 \\ x&: 1 - p_1 = 0 \\ 1&: -p_0 = 0 \end{aligned}\right\} \quad p_0 = 0 \quad p_1 = 1 \quad p_2 = 0 \quad p_3 = -\tfrac{7}{60}$$

$$\therefore \sin(x) \approx \boxed{\frac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2}} = r_{3,3}(x)$$

x	sin(x)	Maclurin	Error	$R_{3,3}(x)$	Error
0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.099833417	0.099833417	-0.000000000	0.099833417	-0.000000000
0.2	0.198669331	0.198669333	-0.000000003	0.198669328	-0.000000003
0.3	0.295520207	0.295520250	-0.000000043	0.295520159	-0.000000047
0.4	0.389418342	0.389418667	-0.000000324	0.389417989	-0.000000353
0.5	0.479425539	0.479427083	-0.000001545	0.479423868	-0.000001670

It appears that a Maclurin does better here

Sect 8.4 # 8c

$$\frac{2x^3 - 3x^2 + 4x - 5}{x^2 + 2x + 4}$$

work:

$$\begin{array}{r} 2x - 7 \\ x^2+2x+4 \,\overline{)\, 2x^3 - 3x^2 + 4x - 5} \\ \underline{2x^3 + 4x^2 + 8x } \\ -7x^2 - 4x - 5 \\ \underline{-7x^2 - 14x - 28} \\ 10x + 23 \end{array}$$

$$\begin{array}{r} x - 0.3 \\ x+2.3 \,\overline{)\, x^2 + 2x + 4} \\ \underline{x^2 + 2.3x } \\ -0.3x + 4 \\ \underline{-0.3x - 0.69} \\ 4.69 \end{array}$$

$$= 2x - 7 + \frac{10x + 23}{x^2 + 2x + 4}$$

$$= 2x - 7 + \frac{10(x + 2.3)}{x^2 + 2x + 4} = \boxed{2x - 7 + \frac{10}{x - 0.3 + \frac{4.69}{x + 2.3}}}$$