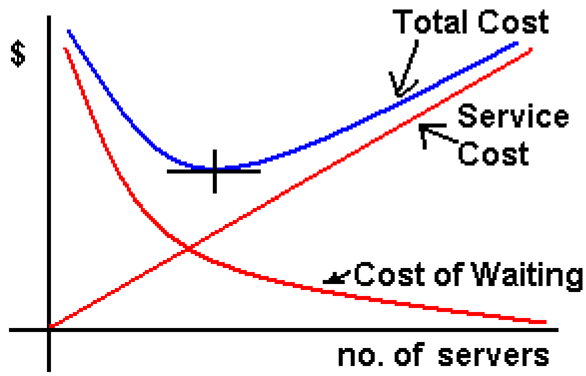
Waiting Lines / Queues **Prof. Richard B. Goldstein**



Arrivals

	simplest model
• One or many?	One
• Independent or dependent?	Independent
• Common inter-arrival distribution?	Yes, exponential distribution.
• Calling population size?	Infinite

$T = t_{i+1} - t_i$ has a common distribution
λ = arrival rate in customers per time period

Service

• One or many?	One
• Independent or dependent?	Independent
• Common service distribution?	Yes, exponential distribution

μ = service rate in customers per time period

Queue Discipline

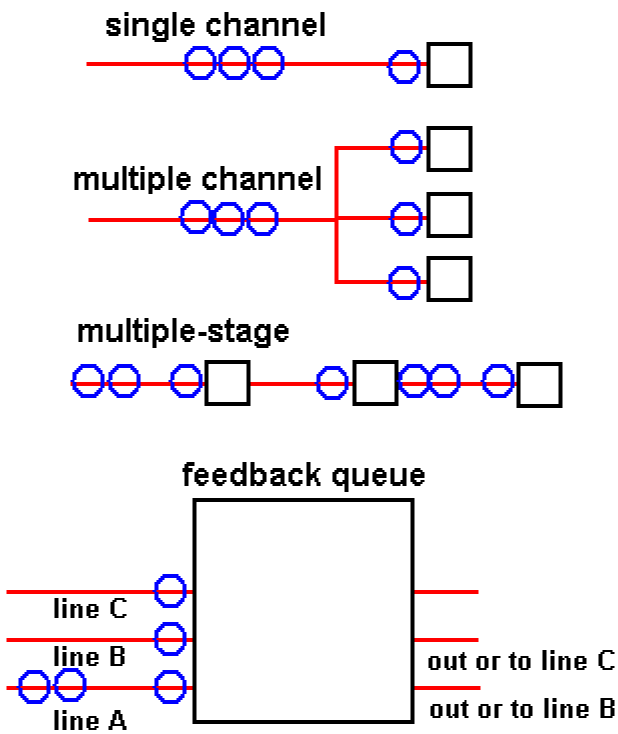
- FIFO or FCFS — Use this one
  first in first out / first come first served
- LIFO - last in first out
- Priority
- Random
- Interrupted Service

Models

• Single Server/Channel	M/M/1 (Arrival/Service/Channels)
• Multiple Server/Channel	M/M/k
• Multiple stage, feedback, etc.	

Kendall's Notation

M = Markovian (no memory exponential), D = deterministic, G = general



Glossary

- **Jockeying** - switching lines before service begins
- **Reneging** - start waiting but leave before service begins
- **Balking** - leave before even getting in line
- **Blocked** - maximum is reached for waiting line and others are not allowed to join

Characteristics

- Probabilities
- Mean number of units and time in the waiting line or system
- Standard Deviation of above
- Busy time of the system

Other models

- Finite calling population
- Fixed or arbitrary service time
- Simulation of more complicated queues

WAITING LINE FORMULAS

λ = arrival rate for customers given in $\left[\dfrac{\textbf{\textit{customers}}}{\textbf{\textit{time unit}}}\right]$

$1/\lambda$ = mean time between two arrivals

μ = service rate for customers in the same units

$1/\mu$ = mean service time for a single customer

Prob {*n* customers arrive in the given time unit} = $\dfrac{\lambda^n e^{-\lambda}}{n!}$

Prob {time between two arrivals $\le$ t} = $1 - e^{-\lambda t}$

Prob {service time $\le$ t} = $1 - e^{-\mu t}$

SINGLE SERVER QUEUE: M/M/1

Utilization Factor (a measure of how busy the system is) = $\rho = \lambda / \mu$

Prob {0 customers in the system} = Prob {system is idle} = $P_0 = 1 - \rho = 1 - \lambda / \mu$

Prob {customer has to wait for service} = $P_W = \lambda / \mu$

Prob {*n* customers in the system} = $P_n = P_0(\rho)^n = (1-\rho)\rho^n$ has a geometric distribution

L = mean number of customers in the system = $\dfrac{\lambda}{\mu - \lambda} = \dfrac{\rho}{1-\rho}$

L_q = mean number of customers in the waiting line (queue) = $\dfrac{\lambda^2}{\mu(\mu-\lambda)} = \dfrac{\rho^2}{1-\rho}$

W = mean time a customer spends in the system = $\dfrac{1}{\mu-\lambda} = \dfrac{L}{\lambda}$

W_q = mean time a customer spends waiting (in the line/queue) = $\dfrac{\lambda}{\mu(\mu-\lambda)} = \dfrac{L_q}{\lambda}$

Note: $W = W_q + \dfrac{1}{\mu}$

MULTIPLE SERVER QUEUE: M/M/k

k = number of channels or servers

Utilization Factor = $\lambda / k\mu$

$$P_0 = \frac{1}{\left[\sum_{n=0}^{k-1} \frac{(\lambda/\mu)^n}{n!}\right] + \frac{(\lambda/\mu)^k}{k!}\left(\frac{k\mu}{k\mu - \lambda}\right)}$$

For example if $k = 3$: $P_0 = \dfrac{1}{1 + \dfrac{\lambda}{\mu} + \dfrac{(\lambda/\mu)^2}{2!} + \dfrac{(\lambda/\mu)^3}{3!}\left(\dfrac{3\mu}{3\mu - \lambda}\right)}$

Continue in this order:

$$L_q = \frac{(\lambda/\mu)^k \lambda\mu}{(k-1)!(k\mu - \lambda)^2} P_0 \qquad L = L_q + \frac{\lambda}{\mu}$$

$$W_q = \frac{L_q}{\lambda} \qquad W = W_q + \frac{1}{\mu}$$

$$P_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{k!k^{n-k}} P_0 & \text{for } n > k \\ \dfrac{(\lambda/\mu)^n}{n!} P_0 & \text{for } 0 \le n \le k \end{cases} \qquad P_w = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \left(\frac{k\mu}{k\mu - \lambda}\right) P_0$$

COSTS

Service Cost = No. of Servers/Channels $*$ Wage of a Server/Channel

The following costs depend upon whether the time waiting in line is important or the total time in the system is important:

Waiting Cost = L_q $*$ Customer Cost

System Cost = L $*$ Customer Cost

TOTAL COST = SERVICE COST + {WAITING or SYSTEM} COST

CLASS example of Waiting Lines

Consider a small grocery with several possible options for serving its customers. Suppose in a typical hour that 24 customers arrive. Cashiers are paid $8/hr, baggers are paid $6/hr, and the time of its valued customers is rated at $4/hr. Cashiers alone can service 30 customers per hour and with a bagger they can service 40 customers per hour. The owners consider the five options:

(A) single cashier
(B) single cashier with a bagger
(C) a single cashier at two different exit doors (parallel lines / no jockeying)
(D) two cashiers with one waiting line
(E) three cashiers with one waiting line

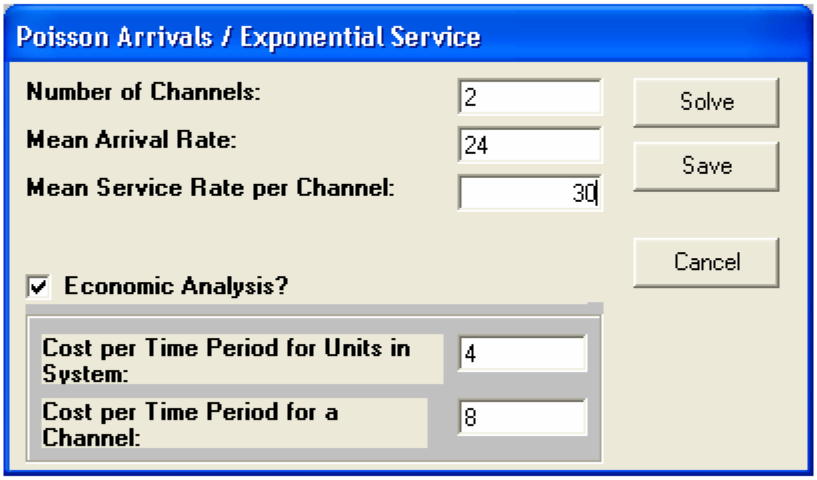
INFORMATION ENTERED:	A	B	C	D	E
Type:	SS	SS	SS	MS	MS
Number of Channels =	1	1	1	2	3
Mean Arrival Rate (λ) =	24	24	12	24	24
Mean Service Rate (μ) =	30	40	30	30	30
Economic Analysis:	Yes	Yes	Yes	Yes	Yes
Cost of Service =	$8	$14	$8	$8	$8
Cost of Waiting =	$4	$4	$4	$4	$4

COMPUTER OUTPUT:

	A	B	C	D	E
Probability of nobody in System (P_0)	20%	40%	20%	42.86%	44.72%
Number of units in queue (line)	3.2	.9	.2667	.1524	.0189
Number of units in system	4	1.5	.6667	.9524	.8189
Time in queue (waiting line)	.1333	.0375	.0222	.0063	.0008
Time in system	.1667	.0625	.0556	.0397	.0341
Utilization rate	80%	60%	40%	40%	26.7%
Prob. an arr. unit will have to wait	80%	60%	40%	22.86%	5.2%
Service cost	$8	$14	$8	$16	$24
Time in the system cost	$16	$6	$2.67	$3.81	$3.28
Total cost	$24	$20	$10.67*	$19.81	$27.28

* This should be doubled for comparison ($21.33)

Waiting Lines - Management Scientist



WAITING LINES
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NUMBER OF CHANNELS = 2
POISSON ARRIVALS WITH MEAN RATE = 24
EXPONENTIAL SERVICE TIMES WITH MEAN RATE = 30 PER CHANNEL
COST FOR UNITS IN THE SYSTEM = $4 PER TIME PERIOD
COST FOR A CHANNEL = $8 PER TIME PERIOD

OPERATING CHARACTERISTICS

THE PROBABILITY OF NO UNITS IN THE SYSTEM	0.4286
THE AVERAGE NUMBER OF UNITS IN THE WAITING LINE	0.1524
THE AVERAGE NUMBER OF UNITS IN THE SYSTEM	0.9524
THE AVERAGE TIME A UNIT SPENDS IN THE WAITING LINE	0.0063
THE AVERAGE TIME A UNIT SPENDS IN THE SYSTEM	0.0397
THE PROBABILITY THAT AN ARRIVING UNIT HAS TO WAIT	0.2286
THE TOTAL COST PER TIME PERIOD	$19.81

Number of Units in the System	Probability
0	0.4286
1	0.3429
2	0.1371
3	0.0549
4	0.0219
5	0.0088
6 OR MORE	0.0059