

# FOURIER SERIES & PARTIAL DIFF. EQS. - Prof. Richard B. Goldstein

**Even function:**  $f(x) = f(-x)$   $b_n = 0 \forall n$ , that is, find  $a_n$  only

**Odd function:**  $f(x) = -f(-x)$   $a_n = 0 \forall n$ , that is, find  $b_n$  only

$$\int f(x) \cos(nx) dx = \frac{f(x) \sin(nx)}{n} + \frac{f'(x) \cos(nx)}{n^2} - \frac{f''(x) \sin(nx)}{n^3} - \dots$$

$$\int f(x) \sin(nx) dx = -\frac{f(x) \cos(nx)}{n} + \frac{f'(x) \sin(nx)}{n^2} + \frac{f''(x) \cos(nx)}{n^3} - \dots$$

$$\cos(n\pi) = (-1)^n \text{ for all integers } n$$

$$\sin(n\pi) = 0 \text{ for all integers } n$$

If  $f(x) = A \cos(nx) + B \sin(mx)$  then  $a_n = A$ ,  $b_m = B$ ,  $a_k = 0$  for  $k \neq n$ , and  $b_k = 0$  for  $k \neq m$

If  $f(x,y) = A \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b})$  on  $[0,a] \times [0,b]$  then  $B_{mn} = A$  and all other  $B_{mn}$ 's = 0

## CLASSIFICATION OF LINEAR SECOND ORDER PDE'S

$$A u_{xx} + 2B u_{xy} + C u_{yy} = F(x,y,u,u_x,u_y)$$

**Elliptic:**  $AC - B^2 > 0$ , **Parabolic:**  $AC - B^2 = 0$ , **Hyperbolic:**  $AC - B^2 < 0$

**WAVE EQUATION:**  $u_{tt} = c^2 \nabla^2 u = c^2(u_{xx} + u_{yy} + \dots)$   $c = \sqrt{\frac{T}{\rho}}$

### STRING ON [0,L]

$$u(x,t) = \sum_1 [B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)] \sin \frac{n\pi x}{L} \text{ where } \lambda_n = \frac{cn\pi}{L}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx, \quad B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin(\frac{n\pi x}{L}) dx \text{ where } f(x) = \text{init. pos. } g(x) = \text{init. vel.}$$

## RECTANGULAR DRUM ON [0,a] x [0,b]

$$u(x,y,t) = \sum_{m=1} \sum_{n=1} (B_{mn} \cos \lambda_{mn} t + B_{mn}^* \sin \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \text{ where } \lambda_{mn} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

$$B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy$$

$$B_{mn}^* = \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy$$

## HEAT EQUATION $u_t = c^2(u_{xx} + u_{yy} + \dots)$ where $c^2 = \frac{k}{\sigma\rho}$

### 1 DIMENSIONAL BAR ON [0,L]

$$u(x,t) = \sum_{n=1} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t} \text{ where } \lambda_n = \frac{cn\pi}{L}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

### 2 DIMENSIONAL RECTANGULAR PLATE ON [0,a] x [0,b] WITH INSULATED (0) ON BOUNDARIES

$$u(x,y,t) = \sum_{m=1} \sum_{n=1} A_{mn} e^{-k\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]t} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$A_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy$$