Numerical Ordinary & Partial Differential Equations - Prof. Richard B. Goldstein

Ordinary Differential Equations

Problem: $y' = f(x, y)$, $y_0 = y(x_0)$

Let $h = x_{n+1} - x_n$, $y_n \approx y(x_n)$

Euler: $y_{n+1} = y_n + hf(x_n, y_n)$ $(n = 0, 1, \ldots)$

Heun (improved Euler):

$k_1 = hf(x_n, y_n)$
$k_2 = hf(x_{n+1}, y_n + k_1)$
$y_{n+1} = y_n + (k_1 + k_2)/2$ $(n = 0, 1, \ldots)$

Runge-Kutta:

$k_1 = hf(x_n, y_n)$
$k_2 = hf(x_n + h/2, y_n + k_1/2)$
$k_3 = hf(x_n + h/2, y_n + k_2/2)$
$k_4 = hf(x_n + h, y_n + k_3)$
$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6$ $(n = 0, 1, \ldots)$

Problem: $y'' = f(x, y, y')$, $y_0 = y(x_0)$, $y'(x_0) = y_0'$

can be converted to a system:

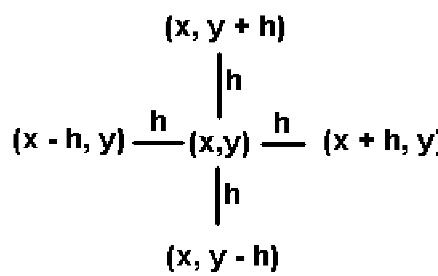
Letting $y_1 = y$ and $y_2 = y'$

$y_1' = y_2$
$y_2' = f(x, y_1, y_2)$

with I.C. $y_1(x_0) = y_0$ and $y_2(x_0) = y_0'$

Partial Differential Equations

Elliptic: Laplace Equation $\nabla^2 u = u_{xx} + u_{yy} = 0$
Poission Equation $\nabla^2 u = u_{xx} + u_{yy} = f(x,y)$



$u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y-h)-4u(x,y) = h^2 f(x,y)$

$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(x_i, y_i)$

In a small mesh this can be solved exactly. In larger meshes Gauss-Seidel approximations will converge to a solution.

Parabolic: Heat Equation

$u_t = u_{xx}$, $u(x, 0) = f(x)$, $u(0, t) = u(1, t) = 0$ for $0 \le x \le 1$, $t \ge 0$

Using a mesh where $u_{ij} = u(x_i, t_j)$ where $h = 1/n$ and $k = \Delta t$

$$\frac{1}{k}(u_{i,j+1} - u_{ij}) = \frac{1}{h^2}(u_{i+1,j} - 2u_{ij} + u_{i-1,j})$$

Letting $r = k/h^2 \le 0.5$,

$$u_{i,j+1} = (1 - 2r)u_{ij} + r(u_{i+1,j} + u_{i-1,j})$$

where $u_{i,0} = f(ih)$, $i = 0, 1, 2, ..., n$

Note: Crank-Nicholson method can be used for increased accuracy.

Hyperbolic: Wave Equation

$u_{tt} = u_{xx}$, $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $u(0, t) = u(1, t) = 0$ for $0 \le x \le 1$, $t \ge 0$

Using a mesh where $u_{ij} = u(x_i, t_j)$ where $h = 1/n$ and $k = \Delta t$

$$\frac{1}{k^2}(u_{i,j+1} - 2u_{ij} + u_{i,j-1}) = \frac{1}{h^2}(u_{i+1,j} - 2u_{ij} + u_{i-1,j})$$

Letting $k = h$ this yields:

$u_{i1} = (u_{i-1,0} + u_{i+1,0}) + kg_i$ (note: usually $g_i = 0$)

$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

Table for Parabolic & Hyperbolic:

t \ x	0	h	2h	...	nh = 1
0	$u_{0,0}$	$u_{1,0}$	$u_{2,0}$	...	$u_{n,0}$
k	$u_{0,1}$	$u_{1,1}$	$u_{2,1}$	...	$u_{n,1}$
2k	$u_{0,2}$	$u_{1,2}$	$u_{2,2}$	...	$u_{n,2}$
...	...	...	...	...	...