Prof. Richard B. Goldstein - LINEAR ALGEBRA

BASIC CONCEPTS

m x n matrix, 1 x n row vector, m x 1 column vector:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ . & \cdots & . \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad a = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$A^T = [a_{ji}]$ (n by m matrix)

Symmetric: $A^T = A$ Skew-Symmetric: $A^T = -A$

Matrix Addition, Scalar Multiplication

$C = A + B$ all matrices are m by n and $c_{ij} = a_{ij} + b_{ij}$
$D = kA$ all matrices are m by n and $d_{ij} = ka_{ij}$

Matrix Multiplication

$C = A \times B$ A is m by n, B is n by p, C = m by p and $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Gauss Elimination

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

is equivalent to $Ax = b$

The augmented matrix is row reduced from $[A \mid b]$ to $[U \mid b']$ by elementary row operations:

(A) interchange two rows
(B) multiplication of a row by a nonzero constant
(C) addition of a constant multiple of one row to another row

Linear Independence, Rank of Matrix

$$c_1\vec{a}_{(1)} + c_2\vec{a}_{(2)} + \ldots + c_n\vec{a}_{(n)} = \vec{0}$$

If the n vector linear combination is equal to the zero vector only when all the c's are zero then this is a set of <u>linearly independent</u> vectors.

The rank of matrix A equals the maximum number of linearly independent columns (or rows) of the matrix A. Use row or column reductions.

Linear Systems: General Properties of Solutions

$Ax = 0$ always has the trivial solution $x_1 = \ldots = x_n = 0$
nontrivial solution exists if $\mathrm{rank}(A) = r < n$

$Ax = b$ if $\mathrm{rank}(A) = \mathrm{rank}(A \mid b) = n$, then there is a unique solution
if $\mathrm{rank}(A) = \mathrm{rank}(A \mid b) < n$, then there infinitely many solution
if $\mathrm{rank}(A) < \mathrm{rank}(A \mid b)$ then there are no solutions

Inverse of a Matrix

$AA^{-1} = A^{-1}A = I_n$ if A is n by n and I_n = n by n identity matrix (1's on diagonal and zero elsewhere)

Find A^{-1} by Gauss Elimination starting with $[A \mid I]$ and ending with $[I \mid A^{-1}]$

Determinants

$\det(A) = a_{11}a_{22} - a_{12}a_{21}$ if A is 2 by 2

$D = \det(A) = a_{1k}C_{1k} + a_{2k}C_{2k} + \ldots + a_{nk}C_{nk}$ expansion kth column

where $C_{jk} = (-1)^{j+k} M_{jk}$ and M_{jk} = determinant of order n - 1 obtained by eliminating the jth row and kth column of A. M_{jk} is called the <u>minor</u> of a_{jk} and C_{jk} is called the <u>cofactor</u> of a_{jk} in D.

Note: (a) interchanging rows (or columns) multiplies the determinant by -1
(b) adding a constant multiple of one row (or column) to another does not change the value of the determinant

Cramer's Rule

If $\mathrm{rank}(A) < n$, then $\det(A) = 0$

If $\det(A) \ldots 0$, then $x_i = D_i/D$
where $D = \det(A)$ and D_i = det(A with the ith column replaced by b)

$A^{-1} = [A_{jk}]^T/\det(A)$ where A_{jk} = the cofactor of a_{jk} in A

Eigenvalues, Eigenvectors

$Ax = \lambda x$ always has the trivial solution $x = 0$

non-trivial solution are found by setting $(A - \lambda I)x = 0$ and finding all eigenvalues first by solving the characteristic polynomial:

$$P_n(\lambda) = \det(A - \lambda I) = 0$$

then solve $(A - \lambda_i I)x_i = 0$ for particular eigenvectors

Special Matrices:

Symmetric	$A^T = A$	
Skew-symmetric	$A^T = - A$	
Orthogonal	$A^T = A^{-1}$	all eigenvalues have $\lvert \lambda_i \rvert = 1$
Hermitian	$\overline{A}^T = A$	have real eigenvalues
Skew-Hermitian	$\overline{A}^T = -A$	have imaginary eigenvalues or zero
Unitary	$\overline{A}^T = A^{-1}$	all eigenvalues have $\lvert \lambda_i \rvert = 1$
Similar Matrices	$B = T^{-1}AT$	B has same eigenvalues as A

Diagonalization

$D = X^{-1}AX$ where X is a matrix with each of its columns another eigenvector

$A = XDX^{-1}$

$D^m = X^{-1}A^mX$

$f(A) = X^{-1}f(D)X$

$P_n(A) = 0$ characteristic polynomial of matrix yields the zero matrix

$Q = x^TAx$ is a quadratic form

Set $X^Tx = y$.

Then $Q = x^TXDX^Tx = y^TDy$ reduces to principal axes.

Single Value Decomposition

$A = UDV^T$ where U and V are orthogonal matrices ($U^TU = V^TV = I$)

A is m by n with $m \$ n$, U is m by m, V is n by n, D is m by n

D given by
$$\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{pmatrix}$$
where $\sigma_1 \$ \sigma_2 \$ \ldots \$ \sigma_n \$ 0$ are the singular values

main use - least square curve fitting - A^TA has eigenvalues