DISTRIBUTIONS RELATED TO THE NORMAL – Prof. R. B. Goldstein

Let X be N(μ, σ). That is, $Z = \frac{X-\mu}{\sigma}$ is N(0, 1)

Then, Z^2 has a χ^2 distribution with 1 degree of freedom

$$P(Z^2 \le x) = F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\sqrt{x}} e^{-t^2/2} dt$$

$$f(x) = F'(x) = \frac{1}{\sqrt{2\pi}} \left[e^{-x/2}\left(\frac{1}{2}x^{-1/2}\right) - e^{-x/2}\left(-\frac{1}{2}x^{-1/2}\right)\right]$$

$$f(x) = \frac{e^{-x/2}}{\sqrt{2\pi x}} \text{ for } 0 \le x < \infty, \text{ a } \chi^2 \text{ distribution with 1 d.f.}$$

The sum (or convolution) of two χ^2 distributions with 1 degree of freedom

$$g(t) = \int_0^t f(x) f(t-x)\,dx = \int_0^t \frac{e^{-x/2}}{\sqrt{2\pi x}} \frac{e^{-(t-x)/2}}{\sqrt{2\pi(t-x)}} dx = \frac{e^{-t/2}}{2\pi} \int_0^t \frac{1}{\sqrt{x(t-x)}} dx$$

$$g(t) = \frac{e^{-t/2}}{2\pi} \int_0^t \frac{1}{\sqrt{\frac{t^2}{4} - \left(x - \frac{t}{2}\right)^2}} dx, \text{ letting } x = \frac{t}{2} + \frac{t}{2}\sin\theta \text{ and } dx = \frac{t}{2}\cos\theta\, d\theta$$

$$g(t) = \frac{e^{-t/2}}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{t\cos\theta/2}{t\cos\theta/2} d\theta = \frac{e^{-t/2}}{2} \text{ for } 0 \le t < \infty, \text{ a } \chi^2 \text{ distribution with 2 d.f.}$$

Note that a χ^2 with 2 d.f. is an exponential distribution and that χ^2 in general is a special case to the gamma distribution.

$$\left(\frac{X_1 - \bar{x}}{\sigma}\right)^2 + \left(\frac{X_2 - \bar{x}}{\sigma}\right)^2 + \cdots + \left(\frac{X_n - \bar{x}}{\sigma}\right)^2 \text{ has a } \chi^2 \text{ distribution with } n-1 \text{ d.f.}$$

$F = \frac{\chi^2_{n-1}/(n-1)}{\chi^2_{m-1}/(m-1)}$ has an F distribution with numerator d.f. = n - 1 and denominator d.f. = m - 1

$T = \frac{Z}{\sqrt{\chi^2_{n-1}/(n-1)}}$ has a Student-t distribution with n – 1 d.f.