FINANCIAL EQUATIONS – Prof. Richard B. Goldstein

Interest is the amount of "rent" one pays for using someone else's money.

SIMPLE INTEREST (one payment of interest)

I = Interest P = principal A = amount t = time (yrs) r = annual rate

$$\boxed{I = Prt} \qquad \text{also } r = \frac{I}{Pt} \text{ and } t = \frac{I}{Pr} \tag{1}$$

$$\boxed{A = P + I = P(1+rt)} \quad \text{also } r = \frac{A-P}{Pt} \text{ and } t = \frac{A-P}{Pr} \tag{2}$$

COMPOUND INTEREST (interest paid on interest)

m = no. of compounding periods per year (ex. quarterly: k = 4, daily k = 365)

$$\boxed{A = P(1+i)^n} \text{ where } i = \frac{r}{k} \text{ and } n = mt. \text{ If } k \to \infty \text{ (continuous) } \boxed{A = Pe^{rt}} \tag{3,4}$$

ANNUITIES (several payments)

If \$P is saved at the end of every year for n years then the future value (FV) is

$$\boxed{FV = P\left[\frac{(1+i)^n - 1}{i}\right]} \qquad \text{Note: one solves for P for a } \textit{sinking fund} \tag{5}$$

AMORTIZATION

If n equal payments of \$PMT with an interest rate of i for each period, then the present value, PV, is given by

$$\boxed{PV = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]} \tag{6}$$

(6)

If a loan of \$PV is taken for n periods with an interest rate of i then each payment is

$$\boxed{PMT = PV\left[\frac{i}{1-(1+i)^{-N}}\right]} \tag{7}$$

and the balance after k payments is given by

$$\boxed{PV(1+i)^k - PMT\left[\frac{(1+i)^k - 1}{i}\right]} \tag{8}$$

ADVANCED

Annuities

Assuming $i = \frac{r}{k}$ and n = kt where k→∞ (continuous) $\boxed{FV = P\left[\frac{e^{rt} - 1}{r}\right]}$ (9)

Now, suppose that the first amount saved is \$P, the second is \$P(1+j), the third is $\$P(1+j)^2$

That is, the future value where $i = \frac{r}{k}$, $j = \frac{g}{k}$, n = kt for annual growth rate, g, is given by

$\boxed{FV = P\left[\frac{(1+i)^n - (1+j)^n}{i - j}\right]}$ and $FV = Pn(1 + i)^{n-1}$ if i = j (10)

which as k→∞ (continuous) $\boxed{FV = P\left[\frac{e^{rt} - e^{gt}}{r - g}\right]}$ (11)

Amortization

Assuming $i = \frac{r}{k}$ and n = kt where k→∞ (continuous) $\boxed{PV = PMT\left[\frac{1 - e^{-rt}}{r}\right]}$ (12)

If the payments increase as above, then $\boxed{PV = PMT\frac{\left[1 - \left(\frac{1+j}{1+i}\right)^n\right]}{i - j}}$ (13)

which as k→∞ (continuous) $\boxed{PV = PMT\left[\frac{1 - e^{-(r-g)t}}{r - g}\right]}$ (14)

RETIREMENT NEEDS

S = current salary
r = rate of return
g = growth rate of salary or payments
t = number of years of working or accumulation
p = proportion of salary saved

$\boxed{FV = pS\left[\frac{e^{rt} - e^{gt}}{r - g}\right]}$ is the future value of one's retirement annuity (15)

where the future salary after n years is $\boxed{Se^{gt}}$ (16)

L = life expectancy when one retires
q = proportion of final salary needed in retirement (which increases each year by g)

To achieve that goal save $\boxed{\frac{q\left(1 - e^{(g-r)L}\right)}{e^{(r-g)t} - 1}}$ of each year's salary. (17)

EXAMPLES

Simple Interest P = \$2,000 with r = 4% = 0.04 and t = 6 months = 0.5 yr

I = (2000)(0.04)(0.5) = \$40 (1)

A = 2,000 + 40 = 2,040 (2)

Compound Interest P = \$2,000 with r = 6% = 0.06 and k = 12 (monthly) for t = 3 yrs

$A = 2000(1+ 0.06/12)^{12(3)} = 2000(1.005)^{36} = \$2,393.36$ (3)

or $A = 2000e^{(0.06)3} = 2000e^{0.18} = \$2,394.43$ if continuous (4)

Annuities

Save \$3,000 for 20 years with r = 6% = 0.06

$$FV = 3000\left[\frac{1.06^{20}-1}{0.06}\right] = 3000(36.78559...) = \$110,356.77 \quad (5)$$

Suppose 3000/12 = \$250 is saved each month with i = 0.06/12 = 0.005, then FV

$$FV = 250\left[\frac{1.005^{240}-1}{0.005}\right] = 250(462.04089...) = \$115,510.22 \quad (5)$$

Amortization

To receive an income of \$12,000 at the end of each year for 20 years with an interest rate, r = 6%, requires

$$PV = P\left[\frac{1-(1+r)^{-n}}{r}\right] = 12000\left[\frac{1-1.06^{-20}}{0.06}\right] = 12000(11.46992..) = \$137,639.05 \text{ today.} \quad (6)$$

To receive an income of \$1,000 at the end of each month for 20 years where i = 6%/12 = 0.005 requires

$$PV = P\left[\frac{1-(1+r)^{-n}}{r}\right] = 1000\left[\frac{1-1.005^{-240}}{0.005}\right] = 1000(139.58077..) = \$139,580.77 \text{ today} \quad (6)$$

A car loan of PV = \$12,000 is taken for 4 years (48 monthly payments) with an interest rate, r = 6% or i = 0.06/12 = 0.005, then the monthly payment is

$$PMT = 12000\left[\frac{0.005}{1-1.005^{-48}}\right] = 12000(0.023485) = \$281.82 \quad (7)$$

and if the car is sold after k = 36 payments then the balance is

$$12000(1.005)^{36} - 281.82\left[\frac{(1.005)^{36}-1}{0.005}\right] = \$14,360.17 - 11,085.70 = \$3,274.47 \quad (8)$$

ADVANCED

Annuity The annuity of (5) above if accumulated continuously gives a FV of

$$FV = 3000\left[\frac{e^{20(0.06)} - 1}{0.06}\right] = 3000(38.6686...) = \$116{,}005.85 \quad (9)$$

Now, suppose that the amount saved increases by g = 3% = 0.03 each year.

$$\text{Then } FV = 3000\left[\frac{(1.06)^{20} - (1.03)^{20}}{0.06 - 0.03}\right] = 3000(46.7008...) = \$140{,}102.42 \quad (10)$$

and if saved monthly,

$$FV = 250\left[\frac{(1.005)^{240} - (1.0025)^{240}}{0.005 - 0.0025}\right] = 250(595.77979...) = \$148{,}944.95 \quad (10)$$

$$\text{and continuously } FV = 3000\left[\frac{e^{20(0.06)} - e^{20(0.03)}}{0.06 - 0.03}\right] = 3000(40.93327...) = \$149{,}799.81 \quad (11)$$

Amortization The amount needed for a continuous stream of $12,000 with an interest rate

r = 6% over a period of 20 years is

$$PV = 12000\left[\frac{1 - e^{-0.06(20)}}{0.06}\right] = 12000(11.64676..) = \$139{,}761.16 \quad (12)$$

The same yearly income of $12,000 with increases yearly of g = 3% ($12,360 in the second year and $12,730.80 in the third year, etc.) requires the PV to be

$$12000\frac{\left[1 - \left(\frac{1.03}{1.06}\right)^{20}\right]}{0.06 - 0.03} = 12000(14.56153...) = \$174{,}738.39 \text{ today and if} \quad (13)$$

$$\text{continuous requires } PV = 12000\left[\frac{1 - e^{-(0.06-0.03)(20)}}{0.06 - 0.03}\right] = 12000(15.0396) = \$180{,}475.35 \quad (14)$$

Retirement Needs After t = 40 years of working and saving p = 15% of one's salary (initially

S=$40,000) which increases by g = 3% each year and the annuity receives r = 7% interest grows to

$$FV = 0.15(40000)\left[\frac{e^{0.07(40)} - e^{0.03(40)}}{0.07 - 0.03}\right] = 6000(328.1132..) = \$1{,}968{,}679.48 \quad (15)$$

with a final salary of $40000e^{(0.03)40}$ = $132,804.68 (16)

To achieve q = 80% of one's salary with a life expectancy of L = 25 years one would

$$\text{need to save } \frac{0.80\left(1 - e^{-(0.07-0.03)25}\right)}{e^{40(0.07-0.03)} - 1} = 0.1279 \text{ or 12.79\% of one's salary each year.} \quad (17)$$