Math 132 – Prof. Richard B. Goldstein – Stewart 6th Ed.–ET – Chap 11

Sect 11.1

#5 $a_n = \dfrac{3(-1)^n}{n!}$ $\quad a_1 = \dfrac{-3}{1} = -3,\ a_2 = \dfrac{3}{2},\ a_3 = \dfrac{-3}{6} = -\dfrac{1}{2},\ a_4 = \dfrac{3}{24} = \dfrac{1}{8},\ a_5 = \dfrac{-3}{120} = -\dfrac{1}{40}$

#13 $\left\{1, -\dfrac{2}{3}, \dfrac{4}{9}, -\dfrac{8}{27}, \dots\right\}$ alternating, multiply by 2, divide by 3 : $\boxed{\dfrac{(-1)^{n+1}2^{n-1}}{3^{n-1}}}$ or $\boxed{\left(-\dfrac{2}{3}\right)^{n-1}}$

#19 $a_n = \dfrac{3+5n^2}{n+n^2}$ $\quad \lim_{n\to\infty} \dfrac{3+5n^2}{n+n^2} \overset{L'H}{=} \lim_{n\to\infty} \dfrac{10n}{1+2n} \overset{L'H}{=} \lim_{n\to\infty} \dfrac{10}{2} = \boxed{5}$

#35 $a_n = \dfrac{\cos^2 n}{2^n}$ $\quad$ since $|\cos^2 n| \le 1 \ \forall n$ and $2^n \to \infty$ $\quad \boxed{a_n \to 0}$ Converges

#67 $\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}}$ $\qquad a_{n+1} = \sqrt{2a_n}$

if this is to converge $a_n \to a$ $\qquad a = \sqrt{2a}$ or $a^2 = 2a$

which implies $a = 0$ or $\boxed{2}$. If $a_n \le 2$, then $2a_n \le 4$ and

the sq root $\sqrt{2a_n} \le \sqrt{4} \le 2$ $\therefore$ $a_{n+1} \le 2$. Hence a_n is bdd. by 2

An increasing bounded series is convergent

Also, $a_n = 2^{1-2^{-n}} \Rightarrow 2^{1-0} = 2$ as $n \to \infty$

Sect 11.2

#3 $\displaystyle\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = -\frac{12}{5} + \frac{12}{25} - \frac{12}{125} + \frac{12}{625} - \dots$

$S_1 = -2.4 = -12/5$
$S_2 = -1.92$
$S_3 = -2.016$
$S_4 = -1.9968$
$S_5 = -2.00064$
$S_6 = -1.999872$
$S_7 = -2.0000256$
$\vdots$
$\Rightarrow \boxed{-2 \text{ conv}}$

#23 $\displaystyle\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ $\quad$ Since $\displaystyle\lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 \ne 0$ the series is $\boxed{\text{divergent}}$

#35 $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \frac{2}{(n+1)(n-1)}$ $\qquad \dfrac{2}{(n+1)(n-1)} = \dfrac{A}{n+1} + \dfrac{B}{n-1}$

$2 = A(n-1) + B(n+1)$
$n=1 \quad 2 = 2B \quad B = 1$
$n=-1 \quad 2 = -2A \quad A = -1$

$\displaystyle = \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{4}} - \tfrac{1}{6}\right) + \left(\cancel{\tfrac{1}{5}} - \tfrac{1}{7}\right) + \dots$

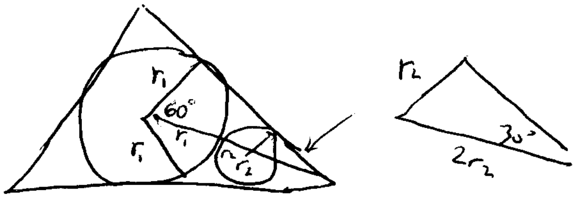
all others drop out

$= 1 + \dfrac{1}{2} = \boxed{\dfrac{3}{2}}$

Sect 11.2

#47 $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ geometric $\quad \left|\frac{x}{3}\right| < 1 \Rightarrow \boxed{|x|<3}$ or $\boxed{-3<x<3}$

#76



$2r_1 = r_1 + r_2 + 2r_2 \quad \therefore \quad r_1 = 3r_2 \quad r_2 = \frac{r_1}{3}$

$A = \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \ldots$

$= \pi r_1^2 + 3\pi r_2^2 \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \ldots\right)$

$= \pi r_1^2 + 3\pi r_2^2 \cdot \left(\frac{1}{1-\frac{1}{9}}\right) = \pi r_1^2 + \frac{27}{8}\pi r_2^2$

$r_1 = \frac{1}{2\sqrt{3}} \quad r_2 = \frac{1}{6\sqrt{3}} \quad \therefore \quad A = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$

triangle area $= \frac{\sqrt{3}}{4} \quad \therefore \quad \boxed{\frac{11\pi/96}{\sqrt{3}/4} \simeq 83.1\%}$

Sect 11.3

#7 $\sum_{n=1}^{\infty} n e^{-n}$ $\quad \int_1^{\infty} x e^{-x}\,dx$ $\quad$ let $u = x$, $dv = e^{-x}dx$, $v = -e^{-x}$, $du = dx$

$-xe^{-x}\Big|_1^{\infty} - \int_1^{\infty} (-e^{-x})\,dx$

$+e^{-1} + \int_1^{\infty} e^{-x}\,dx = e^{-1} + (-e^{-x})\Big|_1^{\infty} = 2e^{-1}$

Since integral exists, the series $\boxed{\text{Converges}}$

#14 $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \ldots = \sum_{n=1}^{\infty} \frac{1}{3n+2}$ $\quad \int_1^{\infty} \frac{1}{3x+2}\,dx = \frac{1}{3}\ln|3x+2|\Big|_1^{\infty} \to \infty$

series $\boxed{\text{diverges}}$

#17 $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ $\quad \int_1^{\infty} \frac{1}{x^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_1^{\infty}$ $\boxed{\text{converges}}$ $\quad$ use $\left(\tan^{-1}(\infty) \to \frac{\pi}{2}\right)$

$\frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]$

#21 $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ $\quad \int_2^{\infty} \frac{1}{x\ln x}\,dx$ $\quad$ let $u = \ln x$, $du = \frac{1}{x}dx$ $\quad \int_{\ln 2}^{\infty} \frac{1}{u}\,du = \ln|u|\Big|_{\ln 2}^{\infty} \to \infty$

$\therefore \boxed{\text{diverges}}$

Sect 11.3

#27 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ if $p=1$ from #21 diverges

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx$ let $u=\ln x$, $du=\frac{1}{x}\,dx$ $\displaystyle\int_{\ln 2}^{\infty} \frac{1}{u^p}\,du$ will converge only if $\boxed{p>1}$

Sect 11.4

#11 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$ $a_n = \frac{n-1}{n4^n}$ $b_n = \frac{1}{4^n}$ $a_n \le b_n$ since $\frac{n-1}{n} \le 1$

$\sum b_n$ is a conv. geom. series $\therefore$ $\boxed{\sum a_n \text{ conv.}}$

#17 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ $a_n = \frac{1}{\sqrt{n^2+1}}$ $b_n = \frac{1}{n}$ diverges

limit comparison $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$ $\therefore$ $\boxed{\text{divergent}}$

#27 $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n} = \sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^2}{e^n}$ let $b_n = \frac{4}{e^n}$ a conv. geom. series

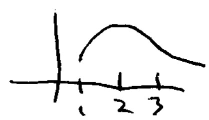
$\displaystyle\frac{\left(1+\frac{1}{n}\right)^2}{e^n} \le \frac{4}{e^n}$ since the max. of numerator is at $n=1$

$\therefore$ $\boxed{\text{convergent}}$

Sect 11.5

#11 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^3+4} \overset{L'H}{=} \lim_{n\to\infty} \frac{2n}{3n^2} = \lim_{n\to\infty} \frac{2}{3n} = 0$

$\displaystyle\frac{d}{dx}\left(\frac{x^2}{x^3+4}\right) = \frac{x(8-x^3)}{(x^3+4)^2} = 0$ at $x=2$ a max. pt.

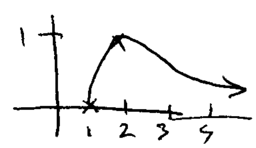


$\therefore$ after $n=2$ decreasing function

$\boxed{\text{Convergent}}$ because $a_n \to 0$ and decreasing after $n=2$

#17 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$ $\displaystyle\lim_{n\to\infty} \left|\sin\left(\frac{\pi}{n}\right)\right| = \sin(0) = 0$

$\sin\left(\frac{\pi}{1}\right) = 0$ $\sin\left(\frac{\pi}{2}\right) = 1$ $\sin\left(\frac{\pi}{3}\right) = 0.866\ldots$ $\sin\left(\frac{\pi}{4}\right) = 0.707\ldots$



as $a_n =$ max at $n=2$ and decreases as $n \to \infty$

$\boxed{\text{Converges}}$

#23 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ $\displaystyle\lim_{n\to\infty} \frac{1}{n^6} = 0$ clearly $\frac{1}{n^6}$ is a decreasing function

$\frac{1}{n^6} = 0.00005 \Rightarrow n^6 = 20{,}000 \Rightarrow n = 5.21\ldots \Rightarrow 6$ is too large and $\frac{1}{6^6}$ too small

the $\boxed{\text{first 5 terms}}$ will be sufficient for the accuracy desired

Sect 11.6

#13 $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{10^{n+1}}{(n+2)4^{2n+3}}}{\dfrac{10^n}{(n+1)4^{2n+1}}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)10}{(n+2)16} \longrightarrow \frac{10}{16} = \frac{5}{8} \text{ as } n\to\infty \quad \therefore \boxed{\text{converges}}$$

(absolutely)

#19 $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$ let $a_n = \frac{1}{n!}$ $\left(\sum \frac{1}{n!} \text{ converges}\right)$

$$b_n = \frac{\cos\left(\frac{n\pi}{3}\right)}{n!}$$

$$\lim_{n\to\infty}\left|\frac{b_n}{a_n}\right| = \lim_{n\to\infty}\left|\cos\left(\frac{n\pi}{3}\right)\right| \le \lim_{n\to\infty} 1 = 1$$

$\therefore$ by comparison $\boxed{\text{converges}}$

(absolutely)

#21 $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ $\quad \sqrt[n]{a_n} = \dfrac{n^2+1}{2n^2+1} \longrightarrow \dfrac{1}{2} < 1$ as $n\to\infty$

$\therefore$ $\boxed{\text{converges}}$ by root test

(absolutely)

Sect 11.7

#11 $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$ Since $\displaystyle\lim_{n\to\infty}\frac{1}{n\ln n} = 0$ $\left(\frac{1}{n\ln n} < \frac{1}{n} \text{ for } n\ge 2\right)$

the series is a $\boxed{\text{convergent alternating series}}$

#15 $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2\cdot5\cdot8\cdots(3n+2)}$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)!}{2\cdot5\cdot8\cdots(3n+2)(3n+5)}}{\dfrac{n!}{2\cdot5\cdot8\cdots(3n+2)}} = \frac{n+1}{3n+5} \to \frac{1}{3}$$

as $n\to\infty$

$\therefore$ $\boxed{\text{Convergent}}$

#27 $\displaystyle\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$

$$\int_1^{\infty}\frac{\ln x}{x^2}\,dx = \left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Bigg|_1^{\infty} \overset{L'H}{=} (0) - (0-1) = 1$$

Converges

Compare to $\sum \frac{\ln n}{n^2} = \sum a_n$

$$\frac{b_k}{a_k} = \frac{\dfrac{k\ln k}{(k+1)^3}}{\dfrac{\ln k}{k^2}} = \frac{k^3}{(k+1)^3} \to 1 \text{ as } k\to\infty$$

$\therefore$ by comparison test $\boxed{\text{Converges}}$

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Sect 11.8

#15 $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(x-2)^{n+1}}{(n+1)^2+1}}{\frac{(x-2)^n}{n^2+1}} = \frac{n^2+1}{(n+1)^2+1}|x-2|$$

$\Rightarrow |x-2| < 1$

$\boxed{R=1}$ center $\boxed{x=2}$

at $x=3$ $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

$x=1$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

both conv.

$\therefore$ $\boxed{[1,3]}$

#17 $\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}}}{\frac{3^n (x+4)^n}{\sqrt{n}}} = \left|\sqrt{\frac{n}{n+1}}\, 3(x+4)\right| \to |3(x+4)| < 1$$

$x+4 < \frac{1}{3}$

$\boxed{R = 1/3}$ center $\boxed{x=-4}$

at $x = -\frac{13}{3}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv. alternating series

at $x = -\frac{11}{3}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent $p=1/2$ series

$\therefore$ $\boxed{\left[-\frac{13}{3}, \frac{11}{3}\right)}$

#23 $\sum_{n=1}^{\infty} n!\,(2x-1)^n$

$$\frac{a_{n+1}}{a_n} = \left|\frac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n}\right| = |(n+1)(2x-1)| \to \infty \quad \forall\ x \text{ except } x = 1/2$$

$\therefore$ $\boxed{R=0}$ only converges for $x = 1/2$

Sect 11.9

#7 $$\frac{x^2}{9+x^2} = \frac{x^2/9}{1+x^2/9} = \frac{x^2}{9}\left(1 - \frac{x^2}{9} + \left(\frac{x^2}{9}\right)^2 - \left(\frac{x^2}{9}\right)^3 + \dots\right) = \frac{x^2}{9} - \left(\frac{x^2}{9}\right)^2 + \left(\frac{x^2}{9}\right)^3 - \left(\frac{x^2}{9}\right)^4 + \dots$$

converges for $\left|\frac{x^2}{9}\right| < 1$ or $-3 < x < 3$

at $x=3$ or -3 $1-1+1-1+\dots$ diverges

$\boxed{(-3,3)}$

#15 $$f(x) = \ln(5-x) \doteq -\int \frac{1}{5-x}dx = -\frac{1}{5}\int \frac{1}{1-\frac{x}{5}}dx = -\frac{1}{5}\int\left(1 + \frac{x}{5} + \frac{x^2}{25} + \frac{x^3}{125} + \dots\right)dx$$

$$= -\frac{1}{5}\left(x + \frac{x^2}{10} + \frac{x^3}{75} + \frac{x^4}{500} + \dots\right) = -\frac{x}{5} - \frac{x^2}{50} - \frac{x^3}{375} - \frac{x^4}{2500} - \dots - \frac{x^n}{n5^n} - \dots$$

$$= \sum_{n=1}^{\infty} -\frac{x^n}{n5^n}$$

$\boxed{|x| < 5 \quad R = 5}$

Converges at $x=-5$ but not at $x=5$

$\boxed{[-5,5)}$

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Sect 11.9

#23 $\int \frac{t}{1-t^8}\,dt = \int t(1+t^8+t^{16}+\dots)\,dt = \int t+t^9+t^{17}+\dots\,dt$

$= \boxed{\frac{t^2}{2}+\frac{t^{10}}{10}+\frac{t^{18}}{18}+\dots}$ $\quad |t|<1 \quad \boxed{R=1}$

Sect 11.10

#5 $f(x)=(1-x)^{-2}$ $\quad f(0)=1$

$f'(x)=-2(1-x)^{-3}(-1)=2(1-x)^{-3}$ $\quad f'(0)=2$

$f''(x)=-6(1-x)^{-4}(-1)=6(1-x)^{-4}$ $\quad f''(0)=6=3!$

$f'''(x)=-24(1-x)^{-5}(-1)=24(1-x)^{-5}$ $\quad f'''(0)=24=4!$

$1+\frac{2!}{1!}x+\frac{3!}{2!}x^2+\frac{4!}{3!}x^3+\dots$

$\boxed{1+2x+3x^2+4x^3+\dots}$ $\quad |x|<1$

#15 $f(x)=e^x$ $\quad a=3$ $\quad f(3)=e^3$

$f'(x)=e^x$ $\quad f'(3)=e^3$ etc.

$\boxed{e^3+\frac{e^3}{1}(x-3)+\frac{e^3}{2!}(x-3)^2+\frac{e^3}{3!}(x-3)^3+\dots}$

converges for $R=\infty$

#23 $f(x)=\sinh x$ $\quad f^{(n+1)}(x)=\begin{cases}\cosh x & n \text{ even}\\ \sinh x & n \text{ odd}\end{cases}$

for $|f^{(n+1)}(x)|$ $\quad |\sinh x|<|\cosh x|=\cosh x$ $\quad \forall x$ $\quad \therefore |f^{(n+1)}(x)|\le\cosh(x)$

$|R_n(x)| \le \frac{\cosh(x)}{(n+1)!}|x|^{n+1}$ for any fixed x as $n\to\infty$ $\quad R_n\to 0$

#33 $f(x)=x\cos\left(\frac{x^2}{2}\right)=x\left[1-\frac{\left(\frac{x^2}{2}\right)^2}{2!}+\frac{\left(\frac{x^2}{2}\right)^4}{4!}-\frac{\left(\frac{x^2}{2}\right)^6}{6!}+\dots\right]$

$= \boxed{x-\frac{x^5}{2^2 2!}+\frac{x^9}{2^4 4!}-\frac{x^{13}}{2^6 6!}+\dots}$

or $\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n}(2n)!}x^{4n+1}, \quad R=\infty}$