

INFINITE SEQUENCES & SERIES – Prof Richard B. Goldstein

SEQUENCES

Definitions/Notation $a_1, a_2, a_3, \dots, a_n, \dots$ or $\{a_n\}$

a sequence $\{a_n\}$ **converges** to the **limit L** if $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow L$ as $n \rightarrow \infty$

(theoretical version) for every $\varepsilon > 0 \exists N$ s.t. $|a_n - L| < \varepsilon$ whenever $n > N$

divergence $a_n \rightarrow \pm\infty$ as $n \rightarrow \infty$ or a_n does not converge to a single L

Properties/laws

$$\left. \begin{array}{l} \{a_n\} \text{ convergent} \\ \{b_n\} \text{ convergent} \\ c \text{ is a constant} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n \\ \lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n \\ \lim_{n \rightarrow \infty} (a_n b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} b_n \right) \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \text{ if denom } \neq 0 \\ \lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p \end{array} \right\}$$

Squeeze Theorem

$$\left\{ \begin{array}{l} a_n \leq b_n \leq c_n \text{ for } n \geq n_0 \\ \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L \end{array} \right\} \Rightarrow \lim_{n \rightarrow \infty} b_n = L$$

Theorem

$$\lim_{n \rightarrow \infty} |a_n| = 0 \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$$

Monotonic – either **increasing** ($a_1 < a_2 < a_3 < \dots \forall n \geq 1$) or **decreasing** ($a_1 > a_2 > a_3 > \dots \forall n \geq 1$)

$\{a_n\}$ is **bounded above** if there is a number M s.t. $a_n \leq M \forall n \geq 1$

$\{a_n\}$ is **bounded below** if there is a number m s.t. $a_n \geq m \forall n \geq 1$

Monotonic Sequence Theorem: Every bounded, monotonic sequence is convergent.

SERIES

$$s_N = \sum_{n=1}^N a_n \text{ (partial summation)} \quad s = \lim_{N \rightarrow \infty} s_N = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \text{ (infinite series)}$$

An infinite series is **convergent** if **s** exists; otherwise it is **divergent**.

TESTS FOR CONVERGENCE OF INFINITE SERIES

Does $\sum_{n=1}^{\infty} a_n$ converge?

(1) If $\lim_{n \rightarrow \infty} a_n$ does not exist or $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series is divergent.

(2) **Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and $a_n = f(n)$.

(A) If $\int_1^{\infty} f(x) dx$ is convergent, then the series converges.

(B) If $\int_1^{\infty} f(x) dx$ is divergent, then the series diverges.

(C) The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(3) **Comparison Tests :** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(A) If $\sum b_n$ is convergent and $a_n \leq b_n \forall n$, then $\sum a_n$ converges.

(B) If $\sum b_n$ is divergent and $a_n \geq b_n \forall n$, then diverges.

(C) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then either both series converge or both diverge.

(D) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

(E) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and diverges, then $\sum a_n$ also diverges.

(4) **Ratio Test:**

(A) If $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, then the series is absolutely convergent.

(B) If $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$, then the series is divergent.

(5) **Root Test :**

(A) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series is absolutely convergent.

(B) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$, then the series is divergent.

(6) **Alternating Series Test:** $\sum_{n=1}^{\infty} (-1)^n b_n = b_1 - b_2 + b_3 - b_4 + \dots$ where $b_n > 0$

If $b_{n+1} \leq b_n \forall n$ and $\lim_{n \rightarrow \infty} b_n = 0$, then the series converges.

ESTIMATIONS

Geometric Series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$ is convergent for $|r| < 1$

$$S_N = \frac{a(1-r^N)}{1-r} \text{ and } S_{\infty} = \frac{a}{1-r}$$

Remainder Estimate for the Integral test:

If $\sum a_n$ converges by the Integral test and $R_n = S - S_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

Alternating Series Estimation Theorem :

$$|R_n| = |S - S_n| \leq b_{n+1}$$

Additional Definitions :

- (1) A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.
- (2) A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Note: if a series is absolutely convergent, then it is convergent