Methods for Solving Differential Equations – Prof. Richard B. Goldstein

Integration

$$\frac{dy}{dx} = y' = f(x) \Rightarrow y = \int f(x)\,dx$$

Example: $y' = 3x^2 + 2,\ y(0) = 4 \Rightarrow y = \int 3x^2 + 2\,dx = x^3 + 2x + C$

where C = 4 after substitution of x = 0 and y = 4

Separable Equations

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \Rightarrow h(y)dy = g(x)dx \Rightarrow \int h(y)\,dy = \int g(x)\,dx$$

Example: $\frac{dy}{dx} = xy + x = x(y+1) \Rightarrow \int \frac{dy}{y+1} = \int x\,dx \Rightarrow \ln(y+1) = \frac{x^2}{2} + C$

$y + 1 = e^{x^2/2 + C} = Ae^{x^2/2} \Rightarrow y = Ae^{x^2/2} - 1$ is the general solution

Constant Coefficient Linear Equations

$$y'' + by' + cy = 0 \Rightarrow \text{let } y = e^{rx} \Rightarrow r^2 + br + c = 0$$

real distinct roots r_1, r_2 : $y = Ae^{r_1 x} + Be^{r_2 x}$

real duplicate root r: $y = (A + Bx)e^{rx}$

complex roots a ± bi $y = e^{ax}(A\cos(bx) + B\sin(bx))$

Examples: $y'' - 5y' + 6y = 0 \Rightarrow y = Ae^{3x} + Be^{2x}$

$y'' + 9y = 0 \Rightarrow y = A\cos(3x) + B\sin(3x)$

Linear Equations

$$y' + P(x)y = Q(x) \Rightarrow \text{Let } I(x) = e^{\int P(x)dx} \Rightarrow y = \frac{\int I(x)Q(x)dx}{I(x)}$$

Example: $y' + \frac{y}{x} = \frac{e^{-x}}{x} \Rightarrow I(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x \Rightarrow y = \frac{\int x\left(\frac{e^{-x}}{x}\right)dx}{x} = \frac{-e^{-x} + C}{x}$

Growth Models

Exponential Growth $\frac{dy}{dt} = ky, y(0) = A \quad \Rightarrow \quad y = Ae^{kt}$

Exponential Decay $\frac{dy}{dt} = -ky, y(0) = A \quad \Rightarrow \quad y = Ae^{-kt}$

Limited Growth $\frac{dy}{dt} = k(L - y), \lim_{t\to\infty} y(t) = L \quad \Rightarrow \quad y = L(1 - e^{-kt})$

Logistic Growth $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right), \lim_{t\to\infty} y(t) = M \Rightarrow \quad y = \frac{M}{1 + Ae^{-kt}}$

Numerical Methods

Euler's Method $y' = F(x, y),\ y_0 = y(x_0)$

$x_1 = x_0 + h,\ y_1 = y_0 + hF(x_0, y_0)$
$x_2 = x_1 + h,\ y_2 = y_1 + hF(x_1, y_1)$
$x_3 = x_2 + h,\ y_3 = y_2 + hF(x_2, y_2)$, etc.

Corrected Euler $y' = F(x, y),\ y_0 = y(x_0)$

$x_1 = x_0 + h,\ p_1 = y_0 + hF(x_0, y_0),\ y_1 = y_0 + 0.5h\{F(x_0, y_0) + F(x_1, p_1)\}$
$x_2 = x_1 + h,\ p_2 = y_1 + hF(x_1, y_1),\ y_2 = y_1 + 0.5h\{F(x_1, y_1) + F(x_2, p_2)\}$
$x_3 = x_0 + h,\ p_3 = y_2 + hF(x_2, y_2),\ y_3 = y_2 + 0.5h\{F(x_2, y_2) + F(x_3, p_3)\}$
etc.

Direction Fields show slopes at various points in a grid of F(x, y)

Example: $F(x, y) = x + y - 1$